Pergamon

J. Appl. Maths Mechs, Vol. 61, No. 2, pp. 265–274, 1997

0021–8928/97 $24.00+0.00

PII: S0021–8928(97)00033–6

A VERSION OF NON-SMOOTH TRANSFORMATIONS FOR ONE-DIMENSIONAL ELASTIC SYSTEMS WITH A PERIODIC STRUCTURE†

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(*Received* 5 *July* 1994)

The method described in [1] of introducing a non-smooth argument by means of special identities is shown to provide an additional means of analysing one-dimensional systems with a periodic structure. A modification of the transformation is constructed which greatly extends the possible applications. © 1997 Elsevier Science Ltd. All rights reserved.

A method for the non-smooth transformations of unknown functions which, in particular, allows the differential equations of systems with rigorous constraints to be written correctly in a suitable form for the use of averaging methods was described in [2, 3].

1. PIECEWISE-LINEAR PERIODIC ARGUMENT AND ASSOCIATED RELATIONS

We will denote by $\tau(x)$ a piecewise-linear 4-periodic sawtooth function (the solid line in Fig. 1), which is defined in a period by the expression

$$\tau(x)=\begin{cases}k_1x, & -(1+\theta)\leqslant x\leqslant 1+\theta\\ k_2(x-2), & 1+\theta\leqslant x<3-\theta\end{cases} \tag{1.1}$$

$$k_1=1/(1+\theta),\quad k_2=-1/(1-\theta);\quad -1\leqslant\theta\leqslant 1$$

where the parameter θ characterizes the slope of the tooth of the saw. Then any $4a$-periodic function $f(x)$ can be represented in the form of a relation which is satisfied for any value of x

$$f(x)=P(\tau)+Q(\tau)\tau',\quad \tau=\tau(x/a) \tag{1.2}$$

where

$$P(\tau)=1/2\ \{(1+\theta)f[(1+\theta)a\tau]+(1-\theta)f[(2-(1-\theta)\tau)a]\}$$
$$Q(\tau)=1/2\ (1-\theta^2)\ \{f[(1+\theta)a\tau]-f[(2-(1-\theta)\tau)a]\} \tag{1.3}$$

The properties of the derivative τ' will be explained below (on the set of isolated points $\{x: \tau(x/a)=\pm 1\}$, the function τ is non-differentiable in the classical sense).

Relation (1.2) can be proved simply by verifying that it is an identity over a period.

Remark. In the non-periodic case there is a similar representation if the function $\tau(x)$ is defined by the relation

$$\tau(x)=\begin{cases}k_1x, & x\geqslant 0\\ k_2x, & x\leqslant 0\end{cases}$$

Taking $\theta=0$ in (1.1), the resulting sawtooth function $\tau(x)$ is symmetric relative to a quarter of a period $x=1$, and the expressions for the functions $P(\tau)$, $Q(\tau)$ are the same as those given in [1].

†*Prikl. Mat. Mekh.* Vol. 61, No. 2, pp. 276–284, 1997.

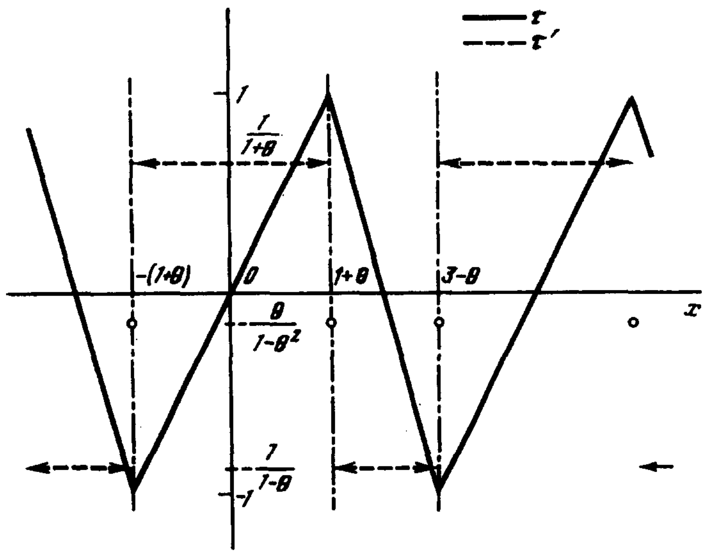

Fig. 1.

Certain properties of the oblique-angled sawtooth function $\tau = \tau(x)$ defined by relation (1.1) should be noted. The square of the derivative τ'^2 for $\theta \neq 0$ is a piecewise-constant function with a periodic series of discontinuities of the first kind, and so is a function of the same class as the derivative τ' itself. Moreover, there is the relation

$$\tau'^2 = \alpha + \beta\tau'; \quad \alpha = 1/(1-\theta^2), \quad \beta = -2\theta/(1-\theta^2) \tag{1.4}$$

which enables the values of the functions τ', τ'^2 to be defined at the actual points of discontinuity.

Thus, putting formally

$$2\tau'\tau'' = (\tau'^2)' = (\alpha + \beta\tau')'$$

we have (see the Remark below)

$$\tau'\tau'' = -\theta(1-\theta^2)^{-1}\tau'' \tag{1.5}$$

whence we find

$$\tau'|_{x=\pm(1+\theta)} = -\theta(1-\theta^2)^{-1} \tag{1.6}$$

Thus, for all x in a period (cf. the dashed line and small circles in Fig. 1), we have

$$\tau'(x) = \begin{cases} 1/(1+\theta), & -(1+\theta) < x < (1+\theta) \\ -\theta/(1-\theta^2), & x = \pm(1+\theta) \\ -1/(1-\theta), & (1+\theta) < x < (3-\theta) \end{cases} \tag{1.7}$$

Remark. We recall that in the theory of distributions, the product of the Dirac δ-function by a function which has a discontinuity at a "point of localization" of a δ-pulse is, generally speaking, undefined. But the co-factors on the left-hand side of relation (1.5) have a singularity of that type. In this case, however, the functions τ', τ'' can be regarded as the limits of sequences generated by one and the same sequence of smooth functions approximating the sawtooth τ. This means that the product τ', τ'' can be given a reasonable interpretation, insofar as its effect on any trial function can be uniquely defined (see [4, 5] in this connection).

Note that in the special case ($\theta = 0$), expressions (1.4)–(1.7) become the corresponding relations for a symmetric sawtooth function [1]. Thus, when $\theta = 0$ we have $\alpha = 1$, $\beta = 0$ and (1.4) takes the form $\tau'^2 = 1$. As a result, the sum $P + Q\tau'$ is an element of the algebra of hyperbolic numbers [6]. When $\theta \neq 0$ relation (1.4) generates an algebra with a more complicated structure.

We shall show that the transformations of the differential equations are greatly simplified as a consequence of the algebraic properties of these expressions.

2. TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS ON A SET OF PERIODIC SOLUTIONS

From the observations made in Section 1, we see that periodic solutions of the differential equations can be sought in the form (1.2). In fact, since the identity (1.2) holds for any periodic function, the required periodic solution can also be represented in form (1.2). We will illustrate this by the example of a linear second-order differential equation of the form

$$a_2(\eta)\frac{d^2u}{d\eta^2}+a_1(\eta)\frac{du}{d\eta}+a_0(\eta)u=q(\eta) \tag{2.1}$$

where $a_i(\eta)$ (i = 0, 1, 2), $q(\eta)$ are continuous periodic functions with period four.

We will seek a twice continuously differentiable solution of Eq. (2.1) in the form

$$u=X(\tau)+Y(\tau)\tau' \tag{2.2}$$

where $\tau=\tau(\eta)$ is the 4-periodic sawtooth function defined by (1.1), the prime denotes the derivative with respect to η; the functions $X(\tau)$, $Y(\tau)$ are to be determined. Formally differentiating Eq. (2.2) with respect to τ and taking into account relation (1.4), we obtain (the dot denotes differentiation with respect to τ)

$$du/d\eta=\alpha\dot{Y}+(\dot{X}+\beta\dot{Y})\tau'+Y\tau'' \tag{2.3}$$

The last term on the right-hand side is a periodic series of δ-pulses which are "localized at points" $\{\eta:\ \tau(\eta)=\pm1\}$. Since the function $u(\eta)$ is continuous, this term is merely formal and can be omitted if we take

$$Y|_{\tau=\pm1}=0 \tag{2.4}$$

Using a similar argument for the second derivative we obtain

$$d^2u/d\eta^2=\alpha\ddot{X}+\alpha\beta\ddot{Y}+[\beta\ddot{X}+(\alpha+\beta^2)\ddot{Y}]\tau' \tag{2.5}$$

under the condition

$$(\dot{X}+\beta\dot{Y})|_{\tau=\pm1}=0 \tag{2.6}$$

The fact that the result of differentiation is an element of the same algebra as the original expression is important.

We will now represent the periodic coefficients and right-hand side of Eq. (2.1) in the form

$$\begin{aligned}a_i(\eta)&=A_i^{(1)}(\tau)+A_i^{(2)}(\tau)\tau' \quad (i=0,1,2)\\ q(\eta)&=Q_1(\tau)+Q_2(\tau)\tau'\end{aligned} \tag{2.7}$$

Using the expressions (2.2)–(2.7) thus obtained, we can transform the original Eq. (2.1) on the set of periodic solutions, including a transition tot a new ("sawtooth") argument τ.

Thus, after we have substituted the corresponding quantities from (2.2)–(2.7) into Eq. (2.1) and taking into account relation (1.4), both sides of the equation will contain two groups of quantities which do and do not contain the factor τ'.

Comparing those groups, we obtain the system of equations

$$\begin{aligned}&\alpha(A_2^{(1)}+\beta A_2^{(2)})\ddot{X}+\alpha(\beta A_2^{(1)}+(\alpha+\beta^2)A_2^{(2)})\ddot{Y}+\\&+\alpha A_1^{(2)}\dot{X}+\alpha(A_1^{(1)}+\beta A_1^{(2)})\dot{Y}+A_0^{(1)}X+\alpha A_0^{(2)}Y=Q_1,\end{aligned} \tag{2.8}$$

$$(\beta A_2^{(1)}+(\alpha+\beta^2)A_2^{(2)})\ddot{X}+((\alpha+\beta^2)A_2^{(1)}+\beta(2\alpha+\beta^2)A_2^{(2)})\ddot{Y}+$$
$$+(A_1^{(1)}+\beta A_1^{(2)})\dot{X}+(\beta A_1^{(1)}+(\alpha+\beta^2)A_1^{(2)})\dot{Y}+A_0^{(2)}X+(A_0^{(1)}+\beta A_0^{(2)})Y=Q_2$$

with boundary conditions (2.4) and (2.6).

The resulting boundary-value problem is formally more complicated than the original equation. In some cases, however, the transformed system has certain advantages. This is especially true of equations in which the coefficients and right-hand side are non-smooth functions which can be expressed simply in terms of the function τ.

To substantiate this, we will give an illustrative example. In Eq. (2.1) let

$$a_2\equiv 1,\ \ a_1\equiv 0,\ \ a_0\equiv 1; \qquad q=Q\tau(\eta)\ \ (Q=\text{const})$$

that is, let the equation have the form

$$d^2u/d\eta^2+u=Q\tau \quad (-\infty<\eta<\infty)$$

In mechanics, this equation can be interpreted as the equation of equilibrium of an infinite string on a linearly-elastic base exposed to a τ-shaped transverse load. For simplicity, we will take $\alpha = 1$, $\beta = 0$ (a symmetric "saw"). In that case boundary-value problem (2.8), (2.4), (2.6) takes the form

$$\ddot{X}+X=Q\tau,\ \ \ddot{Y}+Y=0;\ \ \dot{X}|_{\tau=\pm 1}=0,\ \ Y|_{\tau=\pm 1}=0$$

and has the solution

$$Y\equiv 0,\quad X=Q(\tau-\sin\tau/\cos 1)$$

Thus, the periodic solution has been represented in terms of a standard non-smooth function of quite simple form by means of a single analytic expression, and matching of the solutions at non-smooth points of the external load $\{\eta\colon \tau(\eta) = \pm 1\}$ is "automatic".

Below, however, we make considerable use of a different property of this transformation: the possibility of eliminating space-localized singularities of the periodic structure. In fact, the singular terms which appear as a result of differentiating expression (2.2) can be used to eliminate any periodic singular terms in the original equation. In that case one must proceed from the concept of a weak solution [7], taking the equations in the sense of integral identities.

3. THE PERIODIC PROBLEM FOR AN INFINITE STRING ON LINEARLY-ELASTIC SUPPORTS

To illustrate the approach described in Sections 1 and 2, we will consider the problem of the natural vibrations of an infinite string on linearly-elastic supports arranged periodically. A solution of this problem in the case of equidistant supports was obtained in terms of a sawtooth transformation of the argument in [6]. We assume here that the supports lie in pairs a distance $2(1-\theta)a$ apart, with period $4a$. The equation of the vibrations of the string will have the form

$$\rho\frac{\partial^2 u}{\partial t^2}-T\frac{\partial^2 u}{\partial\eta^2}-\gamma\,\text{sign}\,\tau\tau''u=0 \tag{3.1}$$

where $u = u(t\colon \eta)$ are the coordinates of points of the string, ρ is the density per unit length, T is the tension, 2γ is the average stiffness of the supports over the length of the string and $\tau = \tau(\eta/a)$ is the oblique-angled sawtooth function defined by relation (1.1).

For natural vibrations, putting $u = e^{u\lambda t}U(\eta)$ we obtain a differential equation whose solution can be represented in the form

$$U = X(\tau) + Y(\tau)\tau' \tag{3.2}$$

We determine the derivatives U', U'' using relations (2.3) and (2.5), and after appropriate transformations we arrive at the system of equations

$$\ddot{X}-\frac{2\theta}{1-\theta^2}\ddot{Y}+(1-\theta^2)k^2X=0$$

$$\ddot{Y}-\frac{2\theta(1-\theta^2)}{1+3\theta^2}\ddot{X}+\frac{1+3\theta^2}{(1-\theta^2)^2}k^2Y=0 \quad (k^2=\lambda^2a^2\rho/T) \tag{3.3}$$

with boundary conditions obtained by eliminating the singular terms in the equation

$$Y|_{\tau=\pm1}=0, \quad \left(\dot{X}-\frac{2\theta}{1-\theta^2}\dot{Y}+pX\operatorname{sign}\tau\right)\Big|_{\tau=\pm1}=0 \quad (p=a^2\gamma/T) \tag{3.4}$$

Solving the eigenvalue problem (3.3), (3.4), we determine the required forms of vibrations of the string, that is, the functions $X(\tau)$, $Y(\tau)$. There are four different characteristic forms of vibrations, depending on the relation between the stiffness and the geometric parameters of the string (the quantities k, p and θ).

1. If the parameters k, p and θ are related by the equation

$$\operatorname{tg} k(1+\theta)+\operatorname{tg} k(1-\theta)=2p/k/(1-\theta^2) \tag{3.5}$$

the vibrations (3.2) will have the form

$$X(\tau)=C\left(\cos k(1+\theta)\tau+\frac{1-\theta}{1+\theta}\sigma(\theta,\tau)\right), \quad Y(\tau)=C(1-\theta)(\cos k(1+\theta)\tau-\sigma(\theta,\tau))\times$$

$$\times\left(\sigma(\theta,\tau)=\frac{\cos k(1+\theta)}{\cos k(1-\theta)}\cos k(1-\theta)\tau\right) \tag{3.6}$$

where C is any constant multiplier.

2. If the relation between k, p and θ is described by an equation like (3.5) but with –ctg instead of tg, the eigenfunctions are the same as (3.6) with sin instead of cos.

3. If the supports are so arranged that the distances between them satisfy the relations

$$\frac{2(1-\theta)a}{2(1+\theta)a}=\frac{1-\theta}{1+\theta}=\frac{2n-1}{2m-1} \quad (n,m=1,2,\ldots) \tag{3.7}$$

and

$$k=\frac{\pi(2m-1)}{2(1+\theta)}=\frac{\pi(2n-1)}{2(1-\theta)} \tag{3.8}$$

the forms of the vibrations are obtained in the form (3.6), where $\sigma(\theta,\tau)=(-1)^{m-n+1}\cos k(1-\theta)\tau$.

4. If the ratio of the distances between supports can be put in the form

$$\frac{2(1-\theta)a}{2(1+\theta)a}=\frac{1-\theta}{1+\theta}=\frac{n}{m} \quad (n,m=1,2,\ldots) \tag{3.9}$$

and

$$k=\frac{\pi m}{1+\theta}=\frac{\pi n}{1-\theta} \tag{3.10}$$

the eigenfunctions are obtained from relation (3.6) by replacing cos by sin and $\sigma(\theta,\tau)=(-1)^{m-n+1}\sin k(1-\theta)\tau$.

The results for the special case $\theta=0$ (uniformly situated supports) are the same as in [6].

Figures 2–5 show the characteristic forms of vibrations of a string for the cases described in Paragraphs 1–4. They are constructed in the half-period ($0\leqslant\eta\leqslant2$) and, by symmetry, can be continued periodically over the entire length of the string, as an even function in cases 1 and 3 (Figs 2 and 4) and an odd function in cases 2 and 4 (Figs 3 and 5). The shapes of the graphs were obtained by varying the parameter θ,

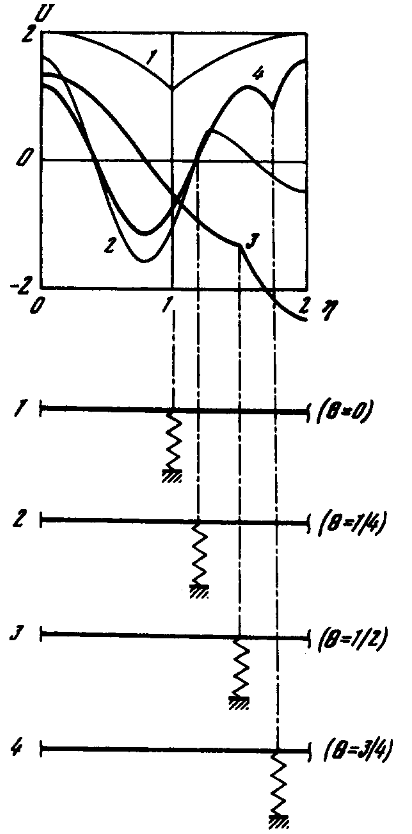

Fig. 2.

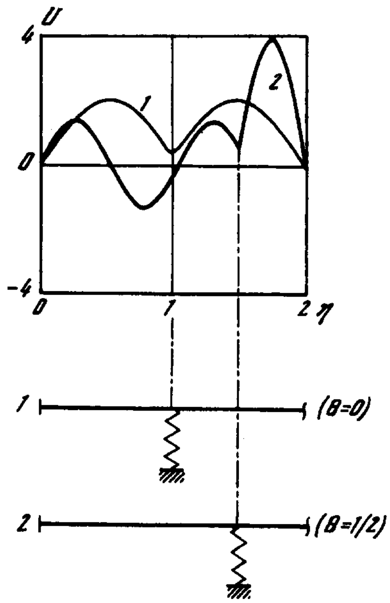

Fig. 3.

that is by altering the position of the supports. Note that the supports are fixed in cases 3 and 4, so that the string does not move at support points. On the other hand, in cases 1 and 2 during vibrations on the whole the supports operate together with the string.

4. THE CASE OF A SLOWLY VARYING LOAD

We will now show that the above technique, together with the idea of averaging, can be used to construct solutions with "fast" (periodic) and "slow" (aperiodic) components. As an example, we will give the solution of the static problem for an infinite string on periodically situated linearly-elastic supports considered above. It will be assumed that the applied load $q = q(\eta^0, \eta^*)$, so that it can be represented as a function of two variables with the two-scales method [3, 8]: a "slow" variable $\eta^0 = \eta$ (for which the previous notation will be used) and a "fast" variable $\eta^* = \eta/\varepsilon$, where ε is a small parameter characterizing the periodicity of the supports. Then the original equation has the form

$$Td^2u/d\eta^2 + \gamma \operatorname{sign} \tau\tau'' u = q(\eta, \eta/\varepsilon) \tag{4.1}$$

Since

$$q = Q_1(\eta, \tau) + Q_2(\eta, \tau)\tau' \tag{4.2}$$

we can represent the required function u in the form (2.2), where $X = X(\eta, \tau)$, $Y = Y(\eta, \tau)$, $\tau = \tau(\eta/\varepsilon)$ is defined by relation (1.1). Differentiating expression (2.2) as a composite function, we obtain

$$d^2u/d\eta^2 = X'' + Y''\tau' + 2\varepsilon^{-1}\left[\Delta(\theta)\dot{Y}' + (\dot{X}' - 2\theta\Delta(\theta)\dot{Y}')\tau'\right] +$$

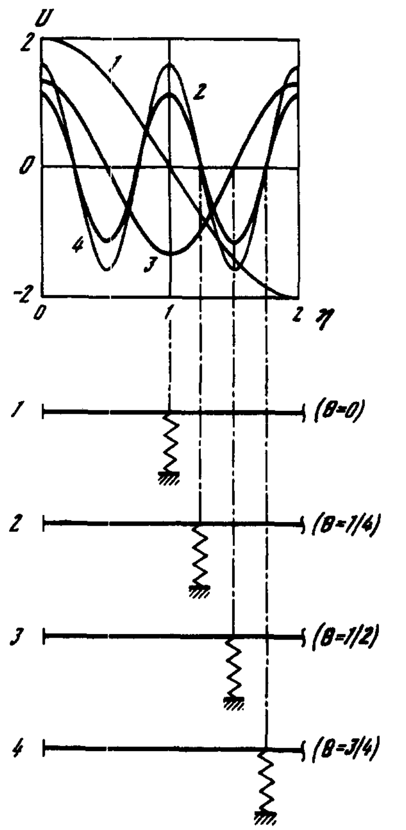

Fig. 4.

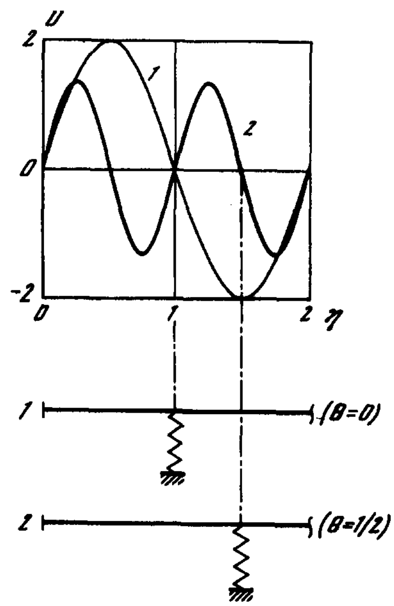

Fig. 5.

$$+\varepsilon^{-2}[\Delta(\theta)(\ddot{X}-2\theta\Delta(\theta)\ddot{Y})+\Delta(\theta)((1+3\theta^2)\Delta(\theta)\ddot{Y}-2\theta\ddot{X})\tau'+$$
$$+(\dot{X}-2\theta\Delta(\theta)\dot{Y})\tau''] \quad (\Delta(\theta)=(1-\theta^2)^{-1}) \tag{4.3}$$

Because relation (4.3) contains a small parameter, it is natural to represent the components of the solution—the functions X and Y—in the form of series in powers of ε

$$X=\sum_{i=0}^{n}\varepsilon^i x_i(\eta,\tau), \quad Y=\sum_{i=0}^{n}\varepsilon^i y_i(\eta,\tau). \tag{4.4}$$

This technique splits the original equation (4.1) into a recurrence sequence of boundary-value problems in the interval $-1\leqslant\tau\leqslant 1$.

1. Equating like expressions in ε^0, we obtain the problem for the functions x_0, y_0

$$\ddot{x}_0-2\theta\Delta(\theta)\ddot{y}_0=0, \quad 2\theta\ddot{x}_0-(1+3\theta^2)\Delta(\theta)\ddot{y}_0=0 \tag{4.5}$$
$$y_0|_{\tau=\pm1}=0, \quad (\dot{x}_0-2\theta\Delta(\theta)\dot{y}_0)|_{\tau=\pm1}=0$$

It follows at once that $x_0=x_0(\eta), y_0\equiv 0$.

2. The problem for the functions x_1, y_1 is similar to (4.5), and thus we have $x_1=x_1(\eta), y_1\equiv 0$. This means that the first two terms in expansions (4.4) are solely slow components of the required solution, while the first fast correction $x_2+y_2\tau'$ is of order ε^2 and is determined from the boundary-value problem

$$-\Delta(\theta)\ddot{x}_2+2\theta\Delta^2(\theta)\ddot{y}_2=x_0''-Q_1(\eta,\tau)/T$$
$$-2\theta\Delta(\theta)\ddot{x}_2+(1+3\theta^2)\Delta^2(\theta)\ddot{y}_2=Q_2(\eta,\tau)/T \tag{4.6}$$

$$y_2|_{\tau=\pm1}=0, \quad (\dot{x}_2-2\theta\Delta(\theta)\dot{y}_2)|_{\tau=\pm1}=\mp(\gamma/T)x_0. \tag{4.7}$$

The solution of this boundary-value problem has the form

$$x_2 = C_1\tau + C_2 - 0,5(1+3\theta^2)x_0''\tau^2 + \frac{\theta}{T}\left[2\Phi(\eta,\tau) + \int \frac{Q_1}{\theta\Delta(\theta)}d\tau\right] \tag{4.8}$$

$$y_2 = C_3\tau + C_4 - \theta\Delta^{-1}(\theta)x_0''\tau^2 + \frac{\Phi(\eta,\tau)}{T\Delta(\theta)} \tag{4.9}$$

where

$$\begin{aligned} C_1 &= -\left[\int Q_1 d\tau|_{\tau=1} + \int Q_1 d\tau|_{\tau=-1} + 2\theta\Delta(\theta)\Phi(\eta,\tau)|_{\tau=-1}^{\tau=1}\right]\frac{1}{2\Delta(\theta)T} \\ C_3 &= -\Phi(\eta,\tau)|_{\tau=-1}^{\tau=1}\frac{1}{2\Delta(\theta)T} \\ C_4 &= \left[\theta x_0'' - \left(\Phi(\eta,\tau)|_{\tau=1} + \Phi(\eta,\tau)|_{\tau=-1}\right)\frac{1}{2T}\right]\frac{1}{\Delta(\theta)} \\ \Phi(\eta,\tau) &= \iint[Q_2(\eta,\tau) + 2\theta Q_1(\eta,\tau)]d\tau d\tau \end{aligned} \tag{4.10}$$

Note that the second relation in boundary conditions (4.7) means that it is possible to find the value of C_1 in the expression for x_2, provided that

$$x_0'' - p^2 x_0 = \frac{1}{2T}\int_{-1}^{1} Q_1 d\tau \quad \left(p^2 = \frac{1}{T}\gamma\Delta(\theta)\right) \tag{4.11}$$

and so actually leads to the averaged equation—the equation for a string on a continuous "smeared" elastic base under the effect of a certain relative load, from which the "slow" function

$$x_0 = A_0 e^{p\eta} + B_0 e^{-p\eta} + x_0^*(\eta) \tag{4.12}$$

is determined, where A_0, B_0 are the constants of integration and $x^*{}_0(\eta)$ is a particular solution of Eq. (4.11).

The "slow" function $C_2 = C_2(\eta)$ of order ε^2 in expression (4.8) is still unknown, and is found from the averaged relation of the ε^4 approximation in the same way as the function x_0. It is not essential to determine this function at this stage, because its contribution to the displacements and stresses is of order ε^2, whereas the corrections introduced into the expressions for the stresses are of order ε when the functions $x_2(\eta, \tau)$, $y_2(\eta, \tau)$ are differentiated with respect to the "fast" variable. There is therefore one more term in (4.4) to be determined—the "slow" function $x_1(\eta)$ of order ε.

3. From the system of equations

$$\begin{aligned} \ddot{x}_3 - 2\theta\Delta(\theta)\ddot{y}_3 &= \Delta^{-1}(\theta)x_1'' - 2\dot{y}_2' \\ 2\theta\Delta(\theta)\ddot{x}_3 - (1+3\theta^2)\Delta^2(\theta)\ddot{y}_3 &= 2\dot{x}_2' - 4\theta\Delta(\theta)\dot{y}_2' \end{aligned} \tag{4.13}$$

with boundary conditions of the form (4.7), we obtain the averaged equation from which to find the function x_1

$$x_1'' - p^2 x_1 = 4\theta x_0''' \tag{4.14}$$

which has a solution analogous to (4.12).

The other terms in series (4.4) are found in the same way, without causing any basic difficulty.

Using expressions (4.8)–(4.10) and (4.12) thus found, we can write the solution of the problem for any given external load q up to terms of order ε inclusive. For example, if we put

$$q = Q_0 \sin\eta, \quad Q_0 = \text{const}$$

to a first approximation we have

$$x_0 = A_0 e^{p\eta} + B_0 e^{-p\eta} + \frac{Q_0}{(1+p^2)T}\sin\eta, \quad y_0 \equiv 0 \tag{4.15}$$

Since we are considering an infinite string, the constants of integration A_0, B_0 in expression (4.15) are equal to zero. The terms of order ε and ε^2 are found in the same way: respectively

$$x_1 \equiv 0, \quad y_1 \equiv 0, \quad x_2 = C_2(\eta) - \frac{(1+3\theta^2)p^2 Q_0}{2(1+p^2)T}\tau^2 \sin\eta$$

$$y_2 = \frac{\theta p^2 Q_0}{(1+p^2)T\Delta(\theta)}(1-\tau^2)\sin\eta \tag{4.16}$$

Expression (4.16) for x_2 includes the unknown "slow" function $C_2(\eta)$, which is found from the averaged equation of order ε^2 in the forms

$$C_2(\eta) = -\frac{(1+3\theta^2)p^2(1+3p^2)Q_0}{6(1+p^2)^2 T}\sin\eta$$

Figures 6 and 7 show graphs of the displacements u (determined up to terms of order ε^2 inclusive) and derivatives $du/d\eta \equiv u'$ (to terms of order ε) with the supports arranged in different ways: $\theta = 0$ and $\theta = 1/2$. The ratio of the structural period 4ε to the period of the external load 2π was taken to be 1/5. All the graphs are drawn in a half-period of the applied load ($0 \leqslant \eta \leqslant \pi$) and can be continued periodically: as an odd function for displacements u and an even function for the derivatives u'.

Note that the accuracy of the solution depends to a considerable extent on the position of the supports, that is, on the value of the parameter θ. Averaging is most effective when the supports are uniformly placed ($\theta = 0$). For limiting values of the parameter $\theta \to \pm 1$ (when the supports lie in pairs, very close to one another) in formula (4.15) $(1 + p^2)^{-1} \ll 1$, and so this quantity is comparable with ε. This causes a loss of accuracy of the asymptotic series (4.4). In such cases, in order to achieve the required accuracy, terms of higher order in series (4.4) must be included.

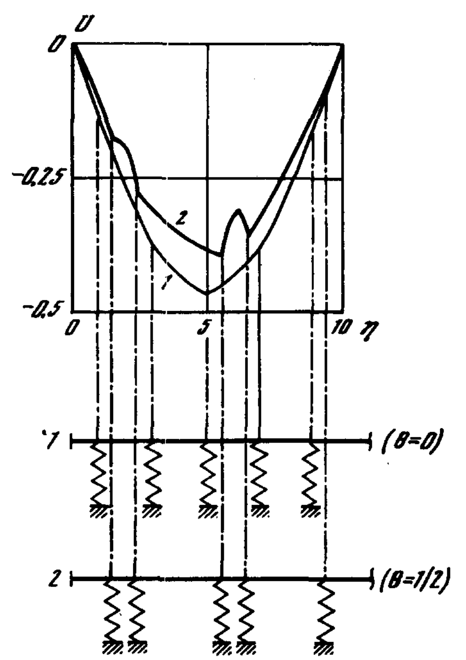

Fig. 6.

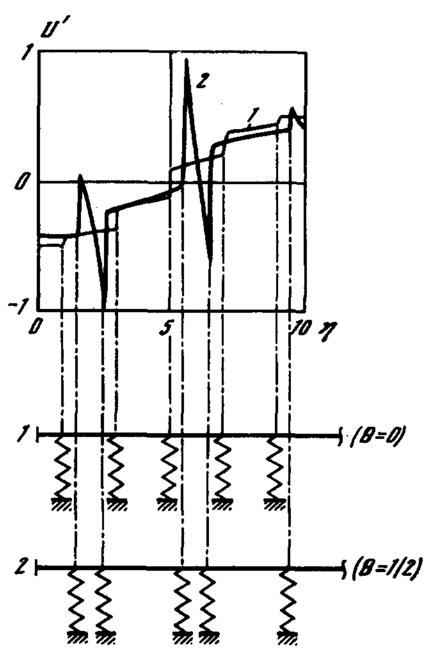

Fig. 7.

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Translated by R.L.